Math 2 HW #2

Key

1 a $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, therefore

$1 = \int_5^{\infty} c\,x^{-3}\,dx$

$= c \cdot 5^{-2}/2$

$c = 50$

b $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$= \int_{-\infty}^{x} \left(\begin{cases} 50\,t^{-3} & \text{if } t > 5 \\ 0 & \text{else} \end{cases} \right) dt$

$= \begin{cases} 0 & \text{if } x < 5 \\ 25(5^{-2} - t^{-2}) & \text{if } x \geq 5 \end{cases}$

c $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$= \int_5^{\infty} x\, 50 x^{-3}\,dx$

$= 50/5 = 10$

d $E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$= \int_5^{\infty} x^2 \cdot 50 x^{-3}\,dx$

$= 50 \int_5^{\infty} x^{-1}\,dx$

It's well known that this integral is infinite.

Since $Var[X] = E[X^2] - E[X]^2$ the variance is infinite.

Since $\sigma[X] = \sqrt{Var[X]}$, the standard deviation is infinite.

2 a $P =$ profit for player B

$H =$ # flips heads

Then $P = n - 2^H$

Let $f(k) = n - 2^k$, then $P = f(H)$

$$
\begin{aligned}
E[P] &= E[f(H)] \\
&= \sum_{k \geq 0} f(k) \cdot P(H = k) \\
&= \sum_{k \geq 0} (n - 2^k)\, 2^{-k-1} \\
&= \underbrace{n\left(\sum_{k \geq 0} 2^{-k-1}\right)}_{= n} - \underbrace{\sum_{k \geq 0} \frac{1}{2}}_{= \infty}
\end{aligned}
$$

So $E[P] = -\infty$.

b $\tilde{P} =$ profit for B under new rules

$$
\begin{aligned}
E(\tilde{P}) &= \sum_{k=0}^{39} (n - 2^k)\, 2^{-k-1} \\
&\quad + f(40)\, P(H = 40) \\
&= n - \left(\sum_{k=0}^{39} \frac{1}{2}\right) - 1 \\
&= n - 21
\end{aligned}
$$

c If you charge $n = \$40$ to play this game and your opponent never flips more than 40 heads in a row, you expect to make \$19 per game!

Nobody has ever flipped 40 heads in a row, so go for it!

3 X = height, has distribution $N[71, 2.5]$. Then

$$\frac{X-71}{2.5} \stackrel{d}{=} N[0,1]$$

Let $Y = \dfrac{X-71}{2.5}$

$$P\{74 \geq X\} = P\left\{\frac{3}{2.5} \geq Y\right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{3/2.5}^{\infty} e^{-y^2/2}\, dy$$
$$\simeq 0.115$$

$$P\{77 \geq X\} = P\left\{\frac{6}{2.5} \geq Y\right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{6/2.5}^{\infty} e^{-y^2/2}\, dy$$
$$\approx 0.0082$$

4 Hazard rate for RV X:

$$\lambda(t) = \frac{f_X(t)}{1-F_X(t)}$$

$X \stackrel{d}{=} \text{Unif}[0,a]$; $\quad f_X = \begin{cases} 1/a & 0 \leq t \leq a \\ 0 & \text{else} \end{cases}$

$$F_X(t) = \begin{cases} 0 & t \leq 0 \\ t/a & 0 \leq t \leq a \\ 1 & t \geq a \end{cases}$$

So

$$\lambda(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ (1/a)/(1-t/a) & \text{if } 0 \leq t \leq a \\ \text{undefined} & \text{if } t \geq a \end{cases}$$

(units of \$1000)

5 Let "b" be the the amount you bid.
Let X (random var) be the amount the other company bids.
Your profit P depends on X; so it is a random var.

$$P = \begin{cases} b-100 & \text{if } b<X \\ 0 & \text{if } b>X \end{cases}$$

We are modeling X as Unif$[70,140]$.
Note that $\{b=X\}$ has probability 0.

$$\begin{aligned} E[P] &= E[P(X)] \\ &= \int_{-\infty}^{\infty} P(x) f_X(x)\,dx \\ &= \int_{70}^{140} \left(\begin{cases} b-100 & \text{if } b<x \\ 0 & \text{else} \end{cases}\right) \frac{1}{70}\,dx \\ &= \int_{\text{Max}(70,b)}^{140} \frac{b-100}{70}\,dx \end{aligned}$$

Suppose we bid between 70 and 140

$$E[P] = \underbrace{(140-b)\frac{(b-100)}{70}}$$

Parabola is concave down;
Max value ½ way between roots

The optimal value of b is thus

$$\frac{140+100}{2} = 120.$$